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On the Existence of Interpolating Projections onto Spline Spaces*

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Sufficient conditions for the existence of a bounded interpolating projection onto subspaces of $C[0, 1]$ are found. For spaces of piecewise polynomial functions the projection can be bounded by the B-spline basis condition number. Infinite interpolation problems are also considered.

1. INTRODUCTION

Let $C[0, 1]$ be the Banach space of continuous functions on the closed interval $[0, 1]$ and let S be a closed subspace. A linear operator P: $C[0, 1] \to S$ is called an *interpolating projection* if there are points $\{t_i\}$ in $[0, 1]$ such that P has the definition: $Pf = s$ if and only if for all t_i, $f(t_i) = s(t_i)$. We establish here some sufficient conditions on S for existence of an interpolating projection onto S and give bounds on the norm of such a projection in terms of the geometry of S. The main corollary is that for spline spaces of fixed degree there is always an interpolating projection whose norm is less than or equal to B-spline basis condition number. This means that the norm of the projection does not depend on the number of knots or their distribution. The proof uses the facts that the B-spline collocation matrix is totally positive and that spline spaces are weak Chebyshev systems. These same properties were used by Goodman and Micchelli [7] recently to prove convergence of interpolating spline functions on a fixed periodic bi-infinite simple knot sequence as the degree of the splines goes to infinity. It seems that the existence of interpolating projections with norm depending on only the local degree of splines was known only for the cases of degree 1 (complexity trivial), degree 2 (Marsden [11]), and degree 3 (de Boor [2]). Most approaches to spline

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interpolation have considered projection operators whose domains consisted of smooth functions or projections onto splines on uniform or quasi-uniform partitions or projections onto low-degree spline spaces. The recent results of Jia [8] to the effect that, for high enough degree splines, interpolation at knot averages gives rise to projections whose norms grow with the number of mesh points (for a geometric mesh) make the results of this paper a bit more interesting than they would otherwise have been. The case of finite meshes is considered in Section 2 and that of infinite meshes in Section 3.

In what follows we assume that S has a (Schauder) basis $\{\phi_i\}$ with the following properties:

(1) there is a number m such that for all sequences $\{\alpha_i\}$

$$m \sup |\alpha_i| \leqslant \left\| \sum \alpha_i \phi_i \right\| \leqslant \sup |\alpha_i|;$$

(2) for every choice of points $\{t_i\}$, the collocation matrices $(\phi_i(t_j))$ are totally non-negative, i.e., all minors are non-negative.

2. Finite-Dimensional Case

We first consider the case where the dimension of S is finite, say, $\dim S = N$. Since P is an interpolating projection with range S, the requirement $P\phi_k = \phi_k$ forces it to have the form

$$(Pf)(x) = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} a_{ij} f(t_j) \right) \phi_i(x)$$

where $\{t_j\}$ are the interpolation points and the a_{ij}'s satisfy $(a_{ij})^{-1} = (\phi_j(t_i))$. By condition (1),

$$\|Pf\| \leqslant \max_i \left| \sum_j a_{ij} f(t_j) \right| \leqslant \|f\| \max_i \sum_j |a_{ij}| \tag{3}$$

and

$$\|Pf\| \geqslant m \cdot \max_i \left| \sum_j a_{ij} f(t_j) \right|.$$

If

$$\max_i \left| \sum a_{ij} f(t_j) \right| = \sum_j |a_{i_0 j}|,$$

then by choosing $\bar{f}$ with $\|\bar{f}\|=1$ and $\bar{f}(t_j)=\operatorname{sgn} a_{i_0 j}$ we get $\|P\| \geqslant \|Pf\| \geqslant m \cdot \max_i \sum_j |a_{ij}|$. Thus, the problem of bounding the norm of P is equivalent, modulo the quantity m, to bounding the l_∞ operator norm of the inverse of the matrix whose (i,j)th entry is $\phi_j(t_i)$. With the l_∞ vector norm $\|x\| := \max |x_i|$ and the associated matrix norm, we recall a result of de Boor [1].

LEMMA 1. *Let A be an $n \times n$ matrix whose $(n-1)\times(n-1)$ principal minors are all non-negative. Suppose there is a vector c such that $(Ac)(i)(-1)^i \geqslant \delta > 0$ for $1 \leqslant i \leqslant n$. Then, A is invertible and $\|A^{-1}\| \leqslant \|c\|/\delta$.*

The key to our analysis is the following result which relies on a theorem of Jones and Karlovitz [9] which was used in a similar way in [7].

LEMMA 2. *Let S be an n-dimensional subspace $C[0,1]$ having a basis $\{\phi_i\}_{i=1}^n$ satisfying* (1) *and* (2). *Then there is $s \in S$ and points $0 \leqslant t_1 < \cdots < t_n \leqslant 1$ such that $s(t_i) = (-1)^i = (-1)^i \|s\|$, $1 \leqslant i \leqslant n$.*

Proof. By (2) $\{\phi_2, \ldots, \phi_n\}$ is a weak Chebyshev system. The Jones–Karlovitz result then says that there exist numbers $\alpha_2, \ldots, \alpha_n$ such that the function $\phi_1 - \sum_{i=2}^n \alpha_i \phi_i$ has points of equioscillation, $0 \leqslant t_1 < \cdots < t_n \leqslant 1$. Multiplying this function by an appropriate number will make it have norm 1. ∎

Remark. This proof replaces an earlier version that (unnecessarily) invoked the Borsuk antipodal theorem. The author thanks Charles Micchelli for pointing out this simplification.

THEOREM 1. *Let S be an n-dimensional subspace of $C[0,1]$ satisfying* (1)–(2), *then there are points $t_1 < \cdots < t_n$ such that the interpolating projection P: $C[0,1] \to S$ determined by these points has norm no greater than m^{-1} (from condition* (1)).

Proof. Let $g \in S$ satisfy: $g(t_i) = (-1)^i = (-1)^i \|g\|$ for some $t_1 < \cdots < t_n$ as guaranteed by Lemma 2. Let $B = (b_{ij}) = (\phi_j(t_i))$ be the corresponding collocation matrix where $\{\phi_i\}$ is the basis for S satisfying (1)–(2). If $g = \sum c_j \phi_j$, then

$$1 = g(t_i)(-1)^i = \sum_j c_j \phi_j(t_i)(-1)^i = (-1)^i (Bc)(t_i).$$

Since $\|c\| \leqslant (1/m)\|g\| = 1/m$, Lemma 1 gives $\|B^{-1}\| \leqslant m^{-1}$. By (3), $\|P\| \leqslant m^{-1}$. ∎

Since the condition number of the L_∞ normalized B-spline basis is $D_{k,\infty} \sim 2^k$ [3], we have

COROLLARY. *Let S be a finite-dimensional polynomial spline space in $C[0, 1]$, then there is an interpolating projection onto S with norm $\leqslant D_{k,\infty}$.*

Remark 1. The use of Chebyshev's theorem makes the preceding argument essentially univariate. The existence of nicely bounded interpolating projections—or, more generally, projections determined by local, positive linear functionals—in the case of multi-dimensional splines appears to be open.

2. We have not prove that the points coming out of Lemma 2 are unique. Nevertheless, Lemma 2 does give a verifiable condition that might lead to a Remez-type algorithm for determining an equioscillating spline on a given mesh.

3. The fact that an interpolating projection onto a spline subspace could be bounded in terms of the (smallest) amplitudes of an oscillating spline function has been known for some time; it was made explicit by de Boor in [2].

4. The nature of the dependence of the points of interpolation on the given knot sequence and degree of the spline space is not revealed by the arguments in this paper. One natural question is whether or not one can choose good points of interpolation by using only local knot averages.

3. SPLINE INTERPOLATION ON BI-INFINITE MESHES

We consider the problem of interpolation of bounded data $\mathbf{y} := \{y_i\}$ by functions of the form

$$\sum_{i=-\infty}^{\infty} \alpha_i N_{i,k}$$

where $\{N_{i,k}\}$ are B-splines of order k on some prescribed bi-infinite mesh. We assume condition (1) holds. In the spirit of the preceding section we show only that it is possible to find points $\{t_i\}$ such that for any given $\mathbf{y} \in l_\infty$ there is a unique $g = \sum \alpha_i N_{i,k}$ with $g(t_i) = y_i$ for all i and $\|g\| \leqslant D_{k,\infty} \|y\|$. Problems of infinite interpolation have been considered by several authors. In particular, both Micchelli's paper [12] and de Boor's [4] discuss their historical antecedents: the work of Schoenberg and Subbotin.

The results of Section 2 insure that for each $M \geqslant 1$, there are points $\{t_i^M : |i| \leqslant M\}$ such that

$$\|(N_{j,k}(t_i^M))^{-1}_{\substack{|i| \leqslant M \\ |j| \leqslant M}}\| \leqslant D_{k,\infty}. \tag{4}$$

By [5], $t_i^M \in \{x: N_{i,k}(x) \geqslant D_{k,\infty}^{-1}\}$. By a diagonal argument we can find $\{t_i: -\infty < i < \infty\}$ and $\{M_l\}$ such that

$$\lim_{l \to \infty} t_i^{M_l} = t_i.$$

By continuity of matrix inversion,

$$\lim_{l \to \infty} \|(N_{j,k}(t_i))^{-1}_{\substack{|i| \leqslant M \\ |j| \leqslant M}} - (N_{j,k}(t_i^{M_l}))^{-1}_{\substack{|i| \leqslant M \\ |j| \leqslant M}}\| = 0$$

for any M. Thus, by (4) the finite sections of $(N_{j,k}(t_i))$ have inverses bounded by $D_{k,\infty}$. Let B_M denote the matrix

$$(N_{j,k}(t_i))_{\substack{|i| \leqslant M \\ |j| \leqslant M}}.$$

By Lemma 1 of [5],

$$\lim_{M \to \infty} B_M^{-1}(i,j) =: C(i,j)$$

exists for every i, j. Furthermore, the uniform (in M) exponential decay of $B_M^{-1}(i,j)$ as $|i-j| \to \infty$ [6] ensures that $\sum_j |C(i,j)|$ converges; therefore, we can assert

$$\begin{aligned} \sum_j |C(i,j)| &= \sum_j \lim_M |B_M^{-1}(i,j)| \\ &= \lim_M \sum_j |B_M^{-1}(i,j)| \leqslant D_{k,\infty}. \end{aligned}$$

Now, for any i, k, $\sum_j C(i,j)\ N_j(t_k) = \sum_j \lim_M B_M^{-1}(i,j)\ N_j(t_k) = \lim_M \sum_j B_M^{-1}(i,j)\ N_j(t_k) = \delta_{i,k}$. Similarly $(N_j(t_i))C = I$. So we have the following from which the interpolation results stated in the first paragraph of this section follow.

PROPOSITION. *Let $\{N_{j,k}\}$ be B-splines of order k on a given bi-infinite mesh. Then there are points $\{t_i\}$ such that $\|(N_{j,k}(t_i))^{-1}\| \leqslant D_{k,\infty}$.*

Finally, it is clear that the arguments used here apply in the semi-infinite case.

Note added in proof. The points satisfying the conclusion of Lemma 2 are unique. This follows from the main result of D. Zwick's paper, "Strong Uniqueness of Best Spline Approximation for a Class of Piecewise n-Convex Functions," to appear.

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